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On the decay of correlations in the generalized spherical model

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Abstract. We point out that the generalized spherical model, as a limit of the isotropic n-vector model, inherits a natural separation into longitudinal and transverse spin projections with respect to the direction of the external field (or spontaneous magnetization), which are distributed in the limit as two independent Gaussian scalar spin systems with different covariance operators. The transverse correlations are well studied within the Berlin–Kac model, which is identified with the corresponding Gaussian model. We consider here the longitudinal correlations and study in detail their spatial fall-off in the transition region. We find a non-trivial temperature dependence, summable clustering away from the critical point (in particular, they obey the fluctuation relation) and a different critical behaviour.

1. Introduction

The spherical model was invented by Berlin and Kac [1] as an approximation to the ferromagnetic Ising model: the Ising spins, $\sigma_x = \pm 1$, are replaced by continuous spins, $S_x \in \mathbb{R}$, subject to a weaker condition, $\sum_{x\in\Lambda} S_x^2 = |\Lambda|$, where Λ is the set of lattice sites. The model is exactly solvable and shows a phase transition for lattice dimension $d \geqslant 3$ with a non-trivial dependence of the critical exponents on d and on the interaction range (see [2], which reviews work up to 1971).

However, the role played by the spherical model in the present theory of phase transitions is rather related to Stanley's remark [3] (proved in the translation-invariant case in [4]) that its free energy is the limit of the (appropriately scaled) isotropic n-vector free energy when $n \to \infty$.

In general, without assuming translation invariance, the n-vector free energy converges to the free energy of a Gaussian model with self-consistently defined covariance, known as the generalized spherical model. More precisely, the latter is defined on the finite set Λ by the Hamiltonian

$$\mathcal{H}_\Lambda = \tfrac{1}{2} \sum_{x,y\in\Lambda} X_{xy} S_x S_y - \sum_{x\in\Lambda} h_x S_x \tag{1.1}$$

where h_x is the external field, $X_{xy} = -J_{xy}$ for $x \neq y$ are the coupling constants, and $X_{xx} = \gamma_x$ are determined as the solutions of the local constraints:

$$\langle S_x^2 \rangle_{\mathcal{H}_\Lambda} = 1 \qquad x \in \Lambda. \tag{1.2}$$

The complete asymptotics of the n-vector model, including that of the equilibrium states, called $1/n$ expansion, is a powerful tool for investigating the critical properties of the model at finite n, and, therefore, has been studied extensively ([5–12] and references therein). Along this line of thought, the generalized spherical model cannot be identified, as done in most papers, with the Gaussian model (1.1) and (1.2). Its equilibrium properties, for example, its 'correlations', should be determined starting with the n-vector model and letting $n \to \infty$ and this provides a far richer picture than that given by the Gaussian equilibrium states. The asymptotics of the n-vector equilibrium states has been fully described in [9] (see also [10, 11], where a more elaborate study for the paramagnetic region is made), and, using different methods, in [12]. The limit of the n-vector state is, indeed, the infinite product of copies of the Gaussian model (1.1) and (1.2), but, when calculating n-vector correlations of physical interest, the higher $1/n$ corrections may sum up to non-trivial contributions in the limit, in qualitative disagreement with the Gaussian result.

The purpose of this paper is to provide a physically significant instance of this phenomenon, not fully recognized until now. A non-zero expectation of the spin in an equilibrium state of the isotropic n-vector model introduces a preferential direction and the spin projections along it or orthogonal to it are expected to behave differently. This is of special interest when there is spontaneous symmetry breaking (spontaneous magnetization), in which case the Goldstone modes play an important destabilizing role in the orthogonal plane, leading to a slow decay of the transverse correlations, while longitudinal correlations are expected to share properties of one-component spin systems (see, for instance, [8, 13]). It is known from standard theories that the $\mathrm{O}(n)$ symmetry induces certain differences between the behaviour of transverse and longitudinal correlations. Both hydrodynamic [14] and microscopic spin-wave approximations [15]—considered to provide accurate results in the limit of long wavelengths and small external magnetic field—lead for fixed $T < T_c$ to a general asymptotic relation between the longitudinal and the transverse correlations of the form

$$\chi_{xy}^{\parallel} = \frac{1}{2\beta m^2}(\chi_{xy}^{\perp})^2 \tag{1.3}$$

where m is the spontaneous magnetization of the system. This shows that the longitudinal fluctuations at long wavelengths are driven by the transverse fluctuations and consequently there should be only one significant correlation length in the system, setting the scale for the decay both for the transverse and longitudinal correlations. A similar result is provided by the renormalization group for the equation of state of an isotropic ferromagnet [16].

Our result in section 3 is essentially a proof of this picture in leading order as $n \to \infty$, i.e. in the generalized spherical model. Indeed, a consequence of the $1/n$ expansion of the n-vector equilibrium state is that the equilibrium distributions of the transverse and longitudinal spin projections have different $n \to \infty$ limits, which turn out to be two independent, but different, Gaussian distributions (section 2). Though the Fourier transforms of the two Gaussian covariance operators (i.e. of the transverse and longitudinal correlations) are known explicitly, the asymptotic relation between the two (below T_c and in zero field) mentioned above is by no means obvious. Its derivation, presented in section 3 shows that it holds due to a singularity developed as $h \searrow 0$, at $k = 0$ in the Fourier transform of $\chi^{\parallel}$, revealed either directly in $\hat{\chi}^{\parallel}(k)$ (for $d \leqslant 4$) or in its derivatives (for $d \geqslant 5$)†.

In section 4 the cross-over from this regime to the critical point asymptotic behaviour as $T \nearrow T_c$, is considered; it exhibits the existence of a specific correlation length for $\chi^{\parallel}$ (the correlation length induced by $\chi^{\perp}$ being infinite) diverging with the same critical exponent as

† For $d \geqslant 5$ though the Fourier transform is finite at $k = 0$, showing that there is no mode softening, the decay of the longitudinal correlations is by no means exponential.

the one from the high-temperature side. This new 'correlation length' is not necessarily defined on the basis of the moments of $\chi^{\parallel}$; it was recognized long ago by Halperin and Hohenberg [17] that it makes sense to speak of a correlation length even in the case when correlations exhibit a power-law decay, as is the case for low-dimensional O(n) symmetric ferromagnets. We stress that for $d = 3$ the behaviour of $\hat{\chi}^{\parallel}(k)$ is similar to that found by Vaks *et al* [15] for a Heisenberg ferromagnet, based on renormalized microscopic theory of spin-waves.

2. Transverse and longitudinal correlations in the generalized spherical model

We consider n-dimensional spins, $\vec{\sigma}_x \in \mathbb{R}^n$, $x \in \Lambda$, living on the finite set Λ, subject to the constraint:

$$\vec{\sigma}_x^2 = n \qquad \forall x \in \Lambda \tag{2.1}$$

a priori distributed with the uniform measure μ on the sphere of radius $\sqrt{n}$, in an external magnetic field along the direction:

$$\vec{e} = \frac{1}{\sqrt{n}}(1, \ldots, 1) \in \mathbb{R}^n. \tag{2.2}$$

Also, we fix some orthogonal direction $\vec{e}_\perp$ $(\vec{e}_\perp^2 = 1, \vec{e}_\perp \cdot \vec{e} = 0)$.

The interaction Hamiltonian is

$$\mathcal{H}_{\Lambda,n} = -\tfrac{1}{2} \sum_{x,y\in\Lambda} J_{xy}\vec{\sigma}_x \cdot \vec{\sigma}_y - \sum_{x\in\Lambda} n^{1/2} h_x \vec{e} \cdot \vec{\sigma}_x \tag{2.3}$$

where $J_\Lambda = (J_{xy})_{x,y\in\Lambda}$ is the coupling-constant matrix, which we suppose to be ferromagnetic $(J_{xy} \geqslant 0, J_{xx} = 0)$ and $h_x \geqslant 0$. We denote by $f_{\Lambda,n}$ the free-energy density

$$f_{\Lambda,n}(\beta, J_\Lambda, h) = -(\beta n|\Lambda|)^{-1} \ln \int \exp(-\beta\mathcal{H}_{\Lambda,n}) \prod_{x\in\Lambda} \mathrm{d}\mu(\vec{\sigma}_x) \tag{2.4}$$

and by $\langle - \rangle_{\Lambda,n}$ the Gibbs state corresponding to $\mathcal{H}_{\Lambda,n}$.

As shown in [9], $f_{\Lambda,n}$, *and all its derivatives*, converge as $n \to \infty$ to the free energy of the Gaussian model (1.1) and (1.2), i.e. to

$$f_\Lambda(\beta, h) = (\beta|\Lambda|)^{-1} \ln \det\left(\frac{\beta}{2\pi} X_\Lambda\right) - (2|\Lambda|)^{-1}\big(h, (X_\Lambda)^{-1}h\big) - (2|\Lambda|)^{-1} \operatorname{tr}(X_\Lambda) \tag{2.5}$$

and, respectively, to its corresponding derivatives, e.g.

$$-|\Lambda| \lim_{n\to\infty} \partial_{h_x} f_{\Lambda,n} = -|\Lambda|\partial_{h_x} f_\Lambda(\beta, h) = [(X_\Lambda)^{-1}h]_x =: (m_\Lambda)_x \tag{2.6}$$

$$-|\Lambda| \lim_{n\to\infty} \partial^2_{h_x h_y} f_{\Lambda,n} = -|\Lambda|\partial^2_{h_x h_y} f_\Lambda(\beta, h) = (X_\Lambda + 2\beta M_\Lambda P_\Lambda M_\Lambda)^{-1}_{xy} =: (\chi^{\parallel}_\Lambda)_{xy}. \tag{2.7}$$

Here, we have denoted by X_Λ the matrix with diagonal elements γ_x, with off-diagonal elements $-J_{xy}$, and satisfying the local conditions (1.2), i.e.

$$\beta^{-1}(X_\Lambda)^{-1}_{xx} = 1 - [(X_\Lambda)^{-1}h]^2_x \qquad x \in \Lambda. \tag{2.8}$$

Also, we introduced the matrices M_Λ and P_Λ defined by

$$M_\Lambda = ((m_\Lambda)_x \delta_{xy})_{x,y\in\Lambda} \qquad (P_\Lambda)^{-1} = ([(X_\Lambda)^{-1}_{xy}]^2)_{x,y\in\Lambda}. \tag{2.9}$$

Note that, due to the h dependence of γ_x, the local susceptibilities $\chi^{\parallel}_{xy}$ do not coincide with the local susceptibilities of the Gaussian model (1.1), i.e. with

$$\beta\langle S_x S_y\rangle^T_{\mathcal{H}_\Lambda} = (X_\Lambda)^{-1}_{xy} =: (\chi^{\perp}_\Lambda)_{xy} \tag{2.10}$$

which is usually taken as the susceptibility of the spherical model. From the results summarized above one sees that, for the generalized spherical model, this is not the only susceptibility worth studying, $\chi^{\|}_{xy}$ being expected to play at least an equal role in the transition. The physical meaning of $\chi^{\|}_{xy}$ and $\chi^{\perp}_{xy}$ is further revealed by considering the equilibrium states.

The equilibrium states $\langle - \rangle_{\Lambda,n}$ converge as $n \to \infty$ to an infinite product of copies of the Gaussian equilibrium state of the model (1.1) and (1.2). In fact, much more is proved in [9]: *all n-vector equilibrium expectations of finite products of spin components have complete asymptotic series in powers of $1/n$*. We write for further reference the first terms of the expansion for the one- and two-spin correlations:

$$\left\langle \sigma^{\mu}_{x} \right\rangle_{\Lambda,n} = \langle S_x \rangle_{\mathcal{H}_\Lambda} - (2n\beta)^{-1} \partial_{h_x} (\log\det H_\Lambda) + \mathrm{O}(n^{-2}) \tag{2.11}$$

$$\left\langle \sigma^{\mu}_{x} \sigma^{\nu}_{y} \right\rangle^{T}_{\Lambda,n} = \delta_{\mu\nu} \langle S_x S_y \rangle_{\mathcal{H}_\Lambda} - (n\beta^2)^{-1} [\delta_{\mu\nu} \partial^2_{h_x h_y} (\log\det H_\Lambda) + (H_\Lambda \partial_{h_x} \gamma_\Lambda, \partial_{h_y} \gamma_\Lambda)] + \mathrm{O}(n^{-2}). \tag{2.12}$$

Here, $\gamma_\Lambda = (\gamma_x(\beta, h))_{x\in\Lambda}$ denotes the diagonal vector of X_Λ and

$$(H_\Lambda)_{xy} = \tfrac{1}{2} ((X_\Lambda)^{-1}_{xy})^2 + \beta((X_\Lambda)^{-1} h)_x (X_\Lambda)^{-1}_{xy} ((X_\Lambda)^{-1} h)_y. \tag{2.13}$$

In this paper, we consider the equilibrium distribution of the projections of $\vec{\sigma}_x$ along the two directions $\vec{e}, \vec{e}_\perp$ introduced above:

$$\xi_x = \vec{e} \cdot \vec{\sigma}_x - \langle \vec{e} \cdot \vec{\sigma}_x \rangle_{\Lambda,n} \qquad \eta_x = \vec{e}_\perp \cdot \vec{\sigma}_x \quad (x \in \Lambda). \tag{2.14}$$

We view $\xi = (\xi_x)_{x\in\Lambda}$, $\eta = (\eta_x)_{x\in\Lambda}$ as vectors in $\mathbb{R}^\Lambda$ and denote by $(\cdot, \cdot)$ the usual scalar product in $\mathbb{R}^\Lambda$. Equation (2.12) allows us to calculate the correlations of ξ, η in the limit

$$\lim_{n\to\infty} \langle \xi_x \xi_y \rangle_{\Lambda,n} = \frac{1}{\beta} (\chi^{\|}_\Lambda)_{xy} \qquad \lim_{n\to\infty} \langle \eta_x \eta_y \rangle_{\Lambda,n} = \frac{1}{\beta} \left(\chi^{\perp}_\Lambda\right)_{xy} \qquad \lim_{n\to\infty} \langle \xi_x \eta_y \rangle_{\Lambda,n} = 0$$

whereby it is seen that the corrections of order $1/n$ sum up in a non-trivial contribution to $\lim_{n\to\infty} \langle \xi_x \xi_y \rangle_{\mathcal{H}_{\Lambda,n}}$.

The following proposition can be easily derived and extends the results of [9], to characteristic functions, i.e. to all correlations.

Proposition 1. *Let $\lambda, \mu \in \mathbb{R}^\Lambda$. Then,*

$$\lim_{n\to\infty} \langle \exp \mathrm{i}[(\lambda, \xi) + (\mu, \eta)] \rangle_{\Lambda,n} = \exp -\frac{1}{2\beta} \left[\left(\lambda, \chi^{\|}_\Lambda \lambda\right) + \left(\mu, \chi^{\perp}_\Lambda \mu\right) \right] \tag{2.15}$$

implying that the random vectors ξ, η converge in the $\langle - \rangle_{\mathcal{H}_{\Lambda,n}}$-distribution to independent Gaussian vectors with covariance matrices $\chi^{\|}_\Lambda$, $\chi^{\perp}_\Lambda$, respectively.

This result shows that the spherical model should be viewed rather as a pair of independent Gaussian models, the Hamiltonian for the η variables being (1.1), and for the ξ variables being $\mathcal{H}^{\|}_\Lambda = \frac{1}{2}\left(\xi, (\chi^{\|}_\Lambda)^{-1} \xi\right)$.

A few remarks are in order.

(a) *Both the ξ and η variables are coupled ferromagnetically*. This is trivial for the η's, as the off-diagonal elements of X_Λ are just the original $-J_{xy}$. For the ξ's this follows from the fact that P_Λ has non-positive off-diagonal elements (it is an M-matrix, cf [9]).

(b) *The fluctuation relation is fulfilled in the generalized spherical model with the fluctuation variables ξ.*

One has to check that $(-1/|\Lambda|)\sum_{x,y}\langle\xi_x\xi_y\rangle_{\mathcal{H}_\Lambda^\parallel}$ equals the second derivative of the free energy with respect to a uniform magnetic field added to the original h. This is obvious, as the ξ's are correlated with the Hessian of $-|\Lambda| f_\Lambda(\beta, h)$. This remark is relevant in connection with the comments (cf, e.g., [18]) concerning the violation of the fluctuation relation in the mean spherical model.

We shall restrict ourselves henceforth to the (physically most interesting) case of spins living on the lattice $\mathbb{Z}^d$ with translation-invariant interactions and external field ($h_x = h$). We take $\Lambda = \Lambda_N = \{0, 1, \ldots, N-1\}^d$ and impose periodic boundary conditions. Thereby, we consider for simplicity nearest-neighbour interactions: $J_{xy} = (2d)^{-1}J$, if x, y are nearest neighbours on the torus Λ_N and $J_{xy} = 0$ otherwise, and take $J = 1$. In view of the uniqueness of the solution of the system (2.8), the spherical fields $\gamma_x^\Lambda(\beta, h)$ are x independent:

$$\gamma_x^\Lambda(\beta, h) = 1 + z_\Lambda(\beta, h) \tag{2.16}$$

and X_Λ is diagonalized by Fourier transformation: denoting $B_N = (2\pi/N)\Lambda_N$, the eigenvalues of X_Λ are $\{z_\Lambda(\beta, h) + \omega(k)\}_{k\in B_N}$, where

$$\omega(k) = 1 - d^{-1}\sum_{\alpha=1}^{d}\cos k^\alpha = \frac{2}{d}\sum_{\alpha=1}^{d}\sin^2\frac{k^\alpha}{2}. \tag{2.17}$$

The system (2.8) reduces to one equation:

$$\beta^{-1}N^{-d}\sum_{k\in B_N}\frac{1}{z+\omega(k)} = 1 - \left(\frac{h}{z}\right)^2 \tag{2.18}$$

which has a unique positive solution $z = z_\Lambda(\beta, h)$. Hence one obtains Fourier representations of all the quantities of interest. We mention, for further reference, that the matrix P_Λ defined in equation (2.9) is diagonal in the same Fourier basis and its eigenvalues, $\{\varphi_\Lambda(k)\}_{k\in B_N}$, are given by the convolution

$$\frac{1}{\varphi_\Lambda(k)} = \frac{1}{N^d}\sum_{q\in B_N}\frac{1}{[z_\Lambda+\omega(q)][z_\Lambda+\omega(k-q)]}. \tag{2.19}$$

We consider now the thermodynamic limit, $N\to\infty$. In the 'regularity region':

$$\{h\neq 0\}\cup\left\{\beta < \beta_c(d) := (2\pi)^{-d}\int_{B^d}\frac{1}{\omega(k)}\,\mathrm{d}k\right\} \tag{2.20}$$

where $B^d = [-\pi,\pi)^d$, the unique solution $z_\Lambda(\beta, h)$ of equation (2.18) is bounded away from zero uniformly in N and converges as $N\to\infty$ to the unique solution $z = z(\beta, h) > 0$ of the limit equation:

$$\beta^{-1}(2\pi)^{-d}\int_{B^d}\frac{1}{z+\omega(k)}\,\mathrm{d}k = 1 - (h/z)^2. \tag{2.21}$$

Hence, the thermodynamic limit is straightforward, and $\chi_\Lambda^\parallel, \chi_\Lambda^\perp$ converge, respectively, uniformly on the compacts of the region (2.20), to the real-analytic functions:

$$\chi_{xy}^\perp = \chi_{xy}^\perp(\beta, h) = \frac{1}{(2\pi)^d}\int_{B^d}\frac{\exp(\mathrm{i}k(x-y))}{z(\beta, h)+\omega(k)}\,\mathrm{d}k \tag{2.22}$$

$$\chi_{xy}^\parallel = \chi_{xy}^\parallel(\beta, h) = \frac{1}{(2\pi)^d}\int_{B^d}\frac{\exp(\mathrm{i}k(x-y))}{z(\beta, h)+\omega(k)+2\beta m^2(\beta, h)\varphi_{z(\beta,h)}(k)}\,\mathrm{d}k \tag{2.23}$$

where

$$\frac{1}{\varphi_z(k)} = \frac{1}{(2\pi)^d}\int_{B^d} \frac{1}{[z+\omega(q)][z+\omega(k-q)]}\,\mathrm{d}q. \tag{2.24}$$

For $d = 1, 2$, $\beta_c(d) = \infty$, and the regularity region (2.20) covers the whole parameter space $\beta > 0, h \in \mathbb{R}$. For $d \geqslant 3$, we have $\beta_c(d) < \infty$, and there will be a phase transition region $\{\beta \geqslant \beta_c(d), h = 0\}$, which we shall approach by taking the limit $h \searrow 0$ in the above expressions. Of course, as seen from equation (2.21), for $\beta > \beta_c(d)$, one has spontaneous magnetization, i.e.

$$m(\beta, h) = h/z(\beta, h) \searrow m(\beta, 0) = \sqrt{1 - \frac{\beta_c(d)}{\beta}}. \tag{2.25}$$

In the regularity region (2.20), $\chi^{\perp}_{xy}$ and $\chi^{\parallel}_{xy}$ have exponential decay, i.e. finite correlation length:

$$\lim_{|x-y|\to\infty} \frac{\left|\ln \chi^{\perp,\parallel}_{xy}(\beta, h)\right|}{|x-y|} = \frac{1}{\lambda^{\perp,\parallel}(\beta, h)} > 0. \tag{2.26}$$

This follows at once from the fact that $\omega(k)$ and $\varphi_z(k)$ (for $z > 0$) are real-analytic functions on the torus B^d. While an explicit formula for $\lambda^{\perp,\parallel}(\beta, h)$ is hard to find in $d > 1$, one can hopefully control their behaviour when (β, h) approaches the transition region. As it happens that $\lambda^{\perp,\parallel} \to \infty$ when approaching in a certain way a given point $(\beta, 0)$, $\beta \geqslant \beta_c(d)$, two problems of physical interest arise:

(a) *the asymptotics of* $\chi^{\perp,\parallel}_{xy}(\beta, 0)$ *i.e. the decay of the correlations in the point* $(\beta, 0)$;
(b) *the possible scaling regimes (continuum limits) around* $(\beta, 0)$.

In the following sections, we consider these problems for $\chi^{\parallel}_{xy}$, as the behaviour of $\chi^{\perp}_{xy}$ is well studied [2].

3. Decay and scaling far from the critical point

In this section we consider the asymptotic behaviour as $x \to \infty, h \searrow 0$ of $\chi^{\parallel}_{0x}(\beta, h)$ for $\beta > \beta_c(d), d \geqslant 3$.

The natural framework for the scaling problem (b) is distribution theory. Let $\mathcal{D}^\circ_d = \mathcal{D}(\mathbb{R}^d\backslash\{0\})$ and $\mathcal{D}^{\circ\prime}_d$ its dual. A function on the lattice $f : \mathbb{Z}^d \to \mathbb{C}$ defines a distribution $\sum_{x\in\mathbb{Z}^d} f(x)\delta_x \in \mathcal{D}^{\circ\prime}_d$, which will also be denoted by f. We adopt the following conventions.

Definition 1. *A family of functions on the lattice, $f(z, \cdot)$, is said to scale in the limit $z \to 0$, if there exists a function $\lambda(z)$ with $\lim_{z\to 0}\lambda(z) = \infty$ and a function F: $\mathbb{R}^d\backslash\{0\} \to \mathbb{R}$, defining a distribution $F \in \mathcal{D}^{\circ\prime}_d$, such that the distributions $\lambda(z)^{-d}\sum_{x\in\mathbb{Z}^d} f(z, x)\delta_{x/\lambda(z)}$ converge in $\mathcal{D}^{\circ\prime}_d$ to F, i.e. if, for all $\phi \in \mathcal{D}^\circ_d$,*

$$\lim_{z\to 0}\lambda(z)^{-d}\sum_{x\in\mathbb{Z}^d} f(z, x)\phi(x/\lambda(z)) = \int_{\mathbb{R}^d} F(x)\,\phi(x)\,\mathrm{d}x. \tag{3.1}$$

The shorthand notation for equation (3.2) is

$$\mathcal{D}^{\circ\prime}_d - \lim_{z\to 0} f(z, x\lambda(z)) = F(x). \tag{3.2}$$

The same frame allows a great simplification in finding the solution to problem (a), also, however, with convergence in a weaker sense. Namely, we adopt the following definition.

Definition 2. *For a function $f:\mathbb{Z}^d\to\mathbb{C}$, we shall say that $\mathcal{D}_d^{\circ\prime}$-$\lim_{x\to\infty} f(x)=L$, if*

$$\lim_{\lambda\to\infty}\lambda^{-d}\sum_{x\in\mathbb{Z}^d} f(x)\phi(x/\lambda)=L\int\phi(x)\,\mathrm{d}x=L(2\pi)^{d/2}\hat{\phi}(0)\qquad\forall\phi\in\mathcal{D}_d^{\circ}. \tag{3.3}$$

The existence of the usual $\lim_{x\to\infty}$ implies, of course, the existence of the $\mathcal{D}_d^{\circ\prime}$-$\lim_{x\to\infty}$ with the same value. The converse is not true without further hypotheses, e.g. $f(x)=(-1)^{x_1}$ has $\mathcal{D}_d^{\circ\prime}$-$\lim_{x\to\infty}$ equal to 0, but no limit. In fact, the above definition implies an averaging process over a 'macroscopic' scale.

The results of this section are contained in the following two propositions, answering the problems (a) and (b), respectively, for $\beta>\beta_c(d)$. For the sake of comparison we also give the analogous results for $\chi_{0x}^{\perp}$.

Proposition 2. *In the sense of definition 2:*

$$\mathcal{D}_d^{\circ\prime}-\lim_{x\to\infty}|x|^{d-2}\chi_{0x}^{\perp}(\beta,0)=K_d \tag{3.4}$$

$$\mathcal{D}_d^{\circ\prime}-\lim_{x\to\infty}|x|^{2d-4}\chi_{0x}^{\parallel}(\beta,0)=\frac{1}{2\beta m(\beta,0)^2}K_d^2 \tag{3.5}$$

where $K_d=\Gamma(d/2-1)/4\pi^{d/2}$.

In considering the scaling limit $h\searrow 0$ at fixed $\beta>\beta_c(d)$, we shall take advantage of the one-to-one correspondence between h and z in the regularity region and of equation (2.25) to take as the scaling parameter z itself. Thereby, we omit mentioning the β dependence and denote $\chi^{\perp,\parallel}(z,x):=\chi_{0x}^{\perp,\parallel}(\beta,h)$.

Proposition 3. *In the sense of definition 1:*

$$\mathcal{D}_d^{\circ\prime}-\lim_{z\searrow 0} z^{(2-d)/2}\chi^{\perp}(z,x/\sqrt{z})=G(x) \tag{3.6}$$

$$\mathcal{D}_d^{\circ\prime}-\lim_{z\searrow 0} z^{2-d}\chi^{\parallel}(z,x/\sqrt{z})=\frac{1}{2\beta m(\beta,0)^2}G(x)^2 \tag{3.7}$$

where $G(x)$ is the Fourier transform of $(2\pi)^{-d/2}\left(1+k^2/2d\right)^{-1}$ (i.e. $G(x-y)$ is the kernel of $(1-(1/2d)\Delta)^{-1}$ in $L^2(\mathbb{R}^d)$).

The proofs of equations (3.5) and (3.7) have, technically, very much in common, so we start by making a few preparatory remarks, useful in both cases.

Both equations (3.5) and (3.7), when compared with (3.4) and (3.6), respectively, say that $\chi^{\parallel}$ has the same behaviour as $(\chi^{\perp})^2/2\beta m(\beta,0)^2$. Upon looking at equations (2.22)–(2.24) and remarking that $1/\varphi_z(k)$ is the Fourier transform of $(\chi^{\perp})^2$, one has in fact to show that $z+\omega(k)$ in the denominator of equation (2.23) is irrelevant in the considered asymptotics. We therefore make the following expansion:

$$\begin{aligned}\frac{1}{z+\omega(k/\lambda)+2\beta m^2\varphi_z(k/\lambda)}&=\frac{1}{2\beta m^2\varphi_z(k/\lambda)}\sum_{j=0}^{s-1}\left[-\frac{z+\omega(k/\lambda)}{2\beta m^2\varphi_z(k/\lambda)}\right]^j\\&\quad+\left[-\frac{z+\omega(k/\lambda)}{2\beta m^2\varphi_z(k/\lambda)}\right]^s\frac{1}{z+\omega(k/\lambda)+2\beta m^2\varphi_z(k/\lambda)}\end{aligned} \tag{3.8}$$

and we have to show that only the $j=0$ term contributes to the limit.

The next remark is related to the $\mathcal{D}_d^{\circ\prime}$-convergence, i.e. with the fact that ϕ in the left-hand side of equations (3.1) and (3.3) has compact support far from 0, implying that all singularities developed (typically, in lower orders in $1/\lambda$) at the origin are wiped out. As we can view the distributions in both sides as being restrictions of distributions in $\mathcal{D}_d'$, we can go to a Fourier representation. Technically, the restriction to $\phi \in \mathcal{D}_d^\circ$ means that all moments of $\hat{\phi}$ vanish, therefore one can subtract at will polynomials (entire functions) from the Fourier transform of the considered distribution without altering its restriction to $\mathcal{D}_d^\circ$. This suggests that it is appropriate to make a Taylor expansion of the Fourier transform. We write down below, for further use, the expansion of $1/\varphi_z(k/\lambda)$. To this end, we introduce the following notation:

$$\omega^{(i)}(q;k) := \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^i [\omega(q-\theta k)]|_{\theta=0} = \begin{cases} \dfrac{(-1)^{(i+1)/2}}{d} \displaystyle\sum_{\alpha=1}^{d} (k_\alpha)^i \sin(q_\alpha) & (i \text{ odd}) \\ \dfrac{(-1)^{(i+2)/2}}{d} \displaystyle\sum_{\alpha=1}^{d} (k_\alpha)^i \cos(q_\alpha) & (i \text{ even}). \end{cases} \tag{3.9}$$

$\omega^{(i)}(q;k)$ are homogeneous polynomials of k, in terms of which one has, for all $z \geqslant 0$, $q \neq 0$:

$$\begin{aligned} \frac{1}{z+\omega(q-k/\lambda)} &= \frac{1}{z+\omega(q)} + \sum_{n=1}^{m-1} \sum_{\substack{p\geqslant 1, i_1,\ldots,i_p \geqslant 1 \\ i_1+\cdots+i_p=n}} \frac{\prod_{j=1}^{p}\left(-\omega^{(i_j)}(q;k/\lambda)/i_j!\right)}{(z+\omega(q))^{p+1}} \\ &\quad + m\int_0^1 \mathrm{d}\theta\,(1-\theta)^{m-1} \sum_{\substack{p\geqslant 1, i_1,\ldots,i_p \geqslant 1 \\ i_1+\cdots+i_p=m}} \frac{\prod_{j=1}^{p}\left(-\omega^{(i_j)}(q-\theta k/\lambda;k/\lambda)/i_j!\right)}{(z+\omega(q-\theta k/\lambda))^{p+1}}. \end{aligned} \tag{3.10}$$

Insertion of equation (3.10) into the definition (2.24) provides a Taylor expansion of $1/\varphi_z(k/\lambda)$, whenever the integrals over q in all terms are convergent. This always happens if $z > 0$. For $z = 0$, the integrability at $q = 0$ is established using the following elementary bounds valid for $q \in B^d$:

$$\omega(q) \geqslant c|q|^2 \qquad \left|\omega^{(i)}(q;k)\right| \leqslant \begin{cases} |k|^i |q| & (i \text{ odd}) \\ |k|^i & (i \text{ even}). \end{cases} \tag{3.11}$$

Finally, one obtains

$$1/\varphi_z(k/\lambda) = P_m(z,k/\lambda) + R_m(z,k/\lambda) \tag{3.12}$$

with $P_m(z,\cdot)$ a polynomial of degree $m-1$, which can be discarded whenever we integrate with a function in $\hat{\mathcal{D}}_d^\circ$; thereby, m is arbitrary, if $z > 0$, but is dimension dependent, if $z = 0$.

With this preparation, we can proceed to the proofs.

Proof of equation (3.5). Applying equation (3.5) to $\phi \in \mathcal{D}_d^\circ$ as in definition 2 and remarking that $|x|^{2d-4}\phi(x) \in \mathcal{D}_d^\circ$ and its Fourier transform is $(-\Delta)^{d-2}\hat{\phi}$, we go to a Fourier transform on the left-hand side, and obtain

$$\lambda^{-d} \sum_{x\in\mathbb{Z}^d} |x|^{2d-4} \chi_{0x}^{\parallel}(\beta,0)\,\phi(x/\lambda) = \frac{\lambda^{d-4}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{(-\Delta)^{d-2}\hat{\phi}(k)}{\omega(k/\lambda) + 2\beta m(\beta,0)^2 \varphi_0(k/\lambda)}\,\mathrm{d}k. \tag{3.13}$$

We have to show that this converges to

$$\frac{1}{2\beta m(\beta,0)^2} K_d^2 (2\pi)^{d/2} \int_{\mathbb{R}^d} \psi_d(k)(-\Delta)^{d-2}\hat{\phi}(k)\,\mathrm{d}k \tag{3.14}$$

where ψ_d is the fundamental solution of $(-\Delta)^{d-2}\psi_d = \delta$ [19]:

$$\psi_d(k) = \begin{cases} A_d |k|^{d-4} & (d \text{ odd}) \\ B_d |k|^{d-4} \ln|k| & (d \text{ even}). \end{cases}$$

With this aim, we use the expansion (3.8) with $z = 0$, and with s chosen so large that the last term can be shown, by simply applying the dominated convergence theorem, to give no contribution to the limit (i.e. to dominate the λ^{d-4} in front of equation (3.13)). This is possible, because $\omega(k)$ is bounded below and above on compacts by $|k|^2$ times positive constants, and thence $1/\varphi_0(k)$ is bounded by $1/|k|$ in $d = 3$, by $|\ln|k||$ in $d = 4$, and by a constant in $d \geqslant 5$. Therefore, one has $s = 1$ for $d = 3, 4$, and $s = \left[\frac{1}{2}d\right] - 1$ for $d \geqslant 5$.

The next step consists in performing the Taylor expansion. We start by considering $1/\varphi_0$. As already remarked, a term $(i_1, \ldots, i_p)$ of the sum in equation (3.10) enters the Taylor polynomial if it provides a convergent integral. A power counting using equation (3.11) shows that this happens whenever $n_{\mathrm{odd}} > 2p - d + 4$, where n_{odd} is the number of odd indices among $(i_1, \ldots, i_p)$. As $n_{\mathrm{even}} + n_{\mathrm{odd}} = p$ and $2n_{\mathrm{even}} + n_{\mathrm{odd}} \leqslant m$, it follows that this happens if $m < d - 4$. Moreover, if d is odd, the $(d-4)$-term vanishes by symmetry. Therefore, by choosing $m = d - 4$, for d even, and $m = d - 3$, for d odd, one can replace $1/\varphi_0(k/\lambda)$ by $R_m(0, k/\lambda)$. Moreover, by the same power counting, all terms $(i_1, \ldots, i_p)$ of $R_m(0, k/\lambda)$, but those with $2n_{\mathrm{even}} + n_{\mathrm{odd}} = m$, lead, by dominated convergence, to polynomials and can therefore be discarded, too. The contribution of $1/\varphi_0$ to the limit of equation (3.13) is thus given by

$$\lim_{\lambda\to\infty} \frac{1}{2\beta m(\beta,0)^2} \lambda^{d-4} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \tilde{R}_m(k/\lambda)(-\Delta)^{d-2}\hat{\phi}(k)\,\mathrm{d}k \tag{3.15}$$

where $\lambda^{d-4}\tilde{R}_m(k/\lambda)$ is (after a translation in the integral over B^d) a sum of terms of the form

$$\begin{aligned} I_p(\lambda, k) &= \lambda^{d-4} \int_0^1 \mathrm{d}\theta\,(1-\theta)^{m-1} \int_{B^d} \frac{\omega^{(1)}(q; k/\lambda)^{2p-m}\omega^{(2)}(q; k/\lambda)^{m-p}}{\omega(q)^{p+1}\omega(q+\theta k/\lambda)}\,\mathrm{d}q \\ &= \int_0^1 \mathrm{d}\theta\,(1-\theta)^{m-1} \int_{B^d} \frac{\omega^{(1)}(q;k)^{2p-m}\omega^{(2)}(q;k)^{m-p}}{\lambda^{m-d+4}\omega(q)^{p+1}\omega(q+\theta k/\lambda)}\,\mathrm{d}q \end{aligned} \tag{3.16}$$

with $m/2 \leqslant p \leqslant m$. The following limits are immediate:

$$\begin{aligned} &\lim_{\lambda\to\infty} \lambda^2 \omega^{(1)}\left(\frac{q}{\lambda}; \frac{k}{\lambda}\right) = -\frac{1}{d} qk = \tilde{\omega}^{(1)}(q, k) \\ &\lim_{\lambda\to\infty} \lambda^2 \omega^{(2)}\left(\frac{q}{\lambda}; \frac{k}{\lambda}\right) = \frac{1}{d} k^2 = \tilde{\omega}^{(2)}(q, k) \end{aligned} \tag{3.17}$$

where $\tilde{\omega}(k) := (1/2d)k^2$ and $\tilde{\omega}^{(i)}$ are defined in terms of $\tilde{\omega}$ by analogy with equation (3.9).

For d odd, after going to the variable $(\lambda/|k|)q$, denoting $\kappa = k/|k|$, and using equation (3.17) in conjunction with equation (3.11), one can apply dominated convergence to show that $I_p(\lambda, k)$ converges for every $k \neq 0$ to $\psi_d(k) = A_d|k|^{d-4}$ times a constant depending on p and d. Namely,

$$\lim_{\lambda\to\infty} I_p(\lambda, k) = |k|^{d-4} 2^{p+2} d^2 \int_0^1 \mathrm{d}\theta\,(1-\theta)^{d-4} \int_{\mathbb{R}^d} \frac{(q\kappa)^{2p-d+3}}{q^{2(p+1)}(q+\theta\kappa)^2}\,\mathrm{d}q. \tag{3.18}$$

The limit in (3.15) follows from dominated convergence, with the bound at ∞ provided by $(-\Delta)^{d-2}\hat{\phi} \in \mathcal{S}(\mathbb{R}^d)$.

The case of even d is slightly trickier. One can verify (using as above the dominated convergence) that, by subtracting

$$J_p(\lambda, k) = \int_0^1 \mathrm{d}\theta\,(1-\theta)^{d-5} \int_{B^d \cap \{|q| \geqslant |k|/\lambda\}} \frac{\omega^{(1)}(q,k)^{2p-d+4}\omega^{(2)}(q,k)^{d-4-p}}{\omega(q)^{p+2}}\,\mathrm{d}q$$

one has $\lim_{\lambda\to\infty}[I_p(\lambda,k) - J_p(\lambda,k)] = C|k|^{d-4}$, which is a polynomial, hence it does not contribute to the limit of (3.15). In turn, $J_p(\lambda,k)$ behaves like

$$2^{p+2}d^2 \int_0^1 \mathrm{d}\theta(1-\theta)^{d-5} \int_{\{1\geqslant|q|\geqslant|k|/\lambda\}} \frac{(qk)^{2p-d+4}k^{2(d-4-p)}}{q^{2(p+2)}}\,\mathrm{d}q = C_p|k|^{d-4}\ln\frac{|k|}{\lambda}$$

in the sense that the difference converges to a polynomial of k, which gives no contribution. Note that, in order to obtain a term proportional to $\psi_d(k) = B_d|k|^{d-4}\ln|k|$, we have to extract here, besides the Taylor polynomial of degree $d-5$, also the diverging term $C_p|k|^{d-4}\ln(1/\lambda)$, which, however, gives zero identically when integrated with $(-\Delta)^{d-2}\hat{\phi}$. For all d the constants C_p in front of $\psi_d(k)$ sum up to provide the value K_d^2 appearing in equation (3.5), but we shall not follow this calculation. □

This finishes the proof of equation (3.5) for $d = 3, 4, 5$. For $d > 5$, we still have to show that the terms $j = 1, \ldots, s-1$ in equation (3.11) do not contribute to the limit. We have shown above that

$$1/\varphi_0(k/\lambda) = \tilde{P}_m(k/\lambda) + \tilde{R}_m(k/\lambda) \tag{3.19}$$

where $\tilde{P}_m$ is a polynomial, eventually including the terms $C_p|k|^{d-4}\ln(1/\lambda)$ subtracted in the case of even d, and $\lim_{\lambda\to\infty}\lambda^{d-4}\tilde{R}_m(k/\lambda) = (2\pi)^d K_d^2\psi_d(k)$. Hence,

$$\frac{\lambda^{d-4}}{\varphi_0(k/\lambda)}\left[\frac{\omega(k/\lambda)}{\varphi_0(k/\lambda)}\right]^j = \lambda^{d-4}\omega(k/\lambda)^j\tilde{P}_m(k/\lambda)^{j+1} + \lambda^{d-4}\tilde{R}_m(k/\lambda)\omega(k/\lambda)^j S(\lambda,k) \tag{3.20}$$

where S contains terms of the form $\tilde{R}_m(k/\lambda)^r\tilde{P}_m(k/\lambda)^{j-r}$ $(r > 0)$. The first term in equation (3.20) vanishes when integrated with $(-\Delta)^{d-2}\hat{\phi}$. For the second term, one uses $\omega(k/\lambda) \leqslant |k|^2/\lambda^2$ and $|P_m(\lambda,k)| \leqslant C(1+|k|)^m\ln\lambda$, to show that it is uniformly polynomially bounded and goes to 0 as $\lambda\to\infty$, and hence the assertion follows.

Proof of equation (3.7). Going to the Fourier transform in the left-hand side and applying definition 1 with $\lambda(z) = 1/\sqrt{z}$, i.e. taking $z = 1/\lambda^2$, we have to calculate, for $\phi\in\mathcal{D}_d^\circ$:

$$\lim_{\lambda\to\infty}\frac{\lambda^{d-4}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{\hat{\phi}(k)}{(1/\lambda)^2+\omega(k/\lambda)+2\beta m(\beta,0)^2\varphi_{1/\lambda^2}(k/\lambda)}\,\mathrm{d}k. \tag{3.21}$$

We proceed as before with the expansion (3.8), followed by the Taylor expansion (3.12). We note that m is arbitrary in equation (3.12), as there are no convergence problems at the origin.

We start by considering the contribution of the first term of equation (3.8). One has to calculate

$$\lim_{\lambda\to\infty}\frac{\lambda^{d-4}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}R_m(1/\lambda^2,k/\lambda)\hat{\phi}(k)\,\mathrm{d}k = \lim_{\lambda\to\infty} m\frac{\lambda^{d-4}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\mathrm{d}k\,\hat{\phi}(k)\int_0^1\mathrm{d}\theta\,(1-\theta)^{m-1}$$
$$\times\sum_{\substack{p\geqslant1,i_1,\ldots,i_p\geqslant1\\ i_1+\cdots+i_p=m}}\int_{B^d}\frac{\prod_{j=1}^p(-\omega^{(i_j)}(q-\theta k/\lambda;k/\lambda)/i_j!)}{(\lambda^{-2}+\omega(q))(\lambda^{-2}+\omega(q-\theta k/\lambda))^{p+1}}\,\mathrm{d}q. \tag{3.22}$$

If $m > d-4$ (to ensure a negative power of λ in front), all terms which lead to convergent q-integrals, or to at most logarithmic divergences, converge to zero. We use the same power counting as in the previous proof to show that this disposes of all terms in the sum with $n_{\mathrm{odd}} \geqslant 2p-d+4$. Let $\tilde{R}_m(1/\lambda^2,k/\lambda)$ denote the sum of the terms with $n_{\mathrm{odd}} < 2p-d+4$. After operating the change of variable $q\to\lambda q-\theta k$ and rearranging the λ-factors in such a way that the pointwise limit of the integrand becomes obvious by use of equation (3.17), we have

$$\lim_{\lambda\to\infty} \lambda^{d-4}\tilde{R}_m(1/\lambda^2, k/\lambda) = \lim_{\lambda\to\infty} m\int_0^1 \mathrm{d}\theta\,(1-\theta)^{m-1} \\ \times \sum \int_{\lambda B^d} \frac{\prod_{j=1}^{p}\left(-\lambda^2\omega^{(i_j)}(q/\lambda; k/\lambda)/i_j!\right)}{(1+\lambda^2\omega((q+\theta k)/\lambda))(1+\lambda^2\omega(q/\lambda))^{p+1}}\,\mathrm{d}q$$

where the sum runs over $p \geqslant 1, i_1, \ldots, i_p \geqslant 1, i_1+\cdots+i_p = m, n_{\mathrm{odd}} < 2p-d+4$. Thereby, we remark that, in view of equation (3.11), only the terms with all $i_j \in \{1,2\}$ have a non-zero pointwise limit. By choosing m sufficiently large to ensure the integrability in q at ∞, one obtains, taking into account that for $i_j > 2$, $\tilde{\omega}^{(i_j)} = 0$:

$$\lim_{\lambda\to\infty} \frac{\lambda^{d-4}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} R_m(1/\lambda^2, k/\lambda)\hat{\phi}(k)\,\mathrm{d}k = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}k\,\hat{\phi}(k) m \int_0^1 \mathrm{d}\theta\,(1-\theta)^{m-1} \\ \times \sum_{\substack{p\geqslant 1, i_1,\ldots,i_p\geqslant 1\\ i_1+\cdots+i_p=m}} \int_{\mathbb{R}^d} \frac{\prod_{j=1}^{p}\left(-\tilde{\omega}^{(i_j)}(q;k)/i_j!\right)}{(1+\tilde{\omega}(q+\theta k))(1+\tilde{\omega}(q))^{p+1}}\,\mathrm{d}q. \qquad (3.23)$$

In fact, an analysis similar to the one given in the proof of equation (3.5) shows that the other terms of equation (3.8) give no contribution, so that equation (3.23) provides the final result for the limit (3.21).

In order to see that this equals the right-hand side of equation (3.7), we have to express the latter as a Fourier transform. As $G(x)^2$ is not locally integrable at 0, a regularization is necessary in order to extend it to a distribution in $\mathcal{D}'_d$, allowing us to use the Fourier representation. For instance, for $\phi \in \mathcal{D}^{\circ}_d$:

$$\int_{\mathbb{R}^d} G(x)^2\phi(x)\,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x)G(y)\frac{\exp\left[-(x+y)^2/2\varepsilon\right]}{(2\pi\varepsilon)^{d/2}}\phi(x)\,\mathrm{d}x\,\mathrm{d}y \\ = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\exp\left(-\varepsilon q^2\right)}{(1+\tilde{\omega}(q))(1+\tilde{\omega}(q-k))}\hat{\phi}(k)\,\mathrm{d}k\,\mathrm{d}q. \qquad (3.24)$$

As $\phi \in \mathcal{D}^{\circ}_d$, we can write the Taylor formula to order m around $k = 0$ and retain only the remainder. By choosing m as before, the integrability at ∞ is achieved, and the application of the dominated convergence theorem yields the right-hand side of equation (3.23). □

Remarks

(a) As already pointed out, proposition 2 states that, in the ordered phase ($\beta > \beta_c, h = 0$), $\chi^{\parallel}_{xy} \sim |x|^{-2(d-2)}$ ($|x-y| \to \infty$). Hence, the correlations of the longitudinal fluctuations decay faster than those of the transverse fluctuations, but still obeying a power law. For $d \geqslant 5$, $\sum_x \chi^{\parallel}_{0x} < \infty$, i.e. the longitudinal fluctuations have summable clustering, leading to normal magnetization fluctuations. In $d = 3$ and 4, $\sum_{|x|<L} \chi^{\parallel}_{0x}$ diverges linearly and, respectively, logarithmically as $L \to \infty$, leading to abnormal magnetization fluctuations.

(b) The scaling relations in proposition 3 control the cross-over from the exponential decay (at $h > 0$) to the power-law decay (at $h = 0$), showing in particular that the correlation lengths $\lambda^{\parallel,\perp} \sim C(\beta)^{\parallel,\perp}/\sqrt{h}$ for $h \searrow 0$, $\beta > \beta_c$, where $C(\beta)^{\perp} = 2C(\beta)^{\parallel}$.

(c) The Fourier transforms of $\chi^{\parallel,\perp}_{0x}$ equal, respectively, by equations (2.22) and (2.23), the inverses of the dispersion laws of the corresponding Gaussian models. The divergence of $\lambda^{\perp}$ as $h \searrow 0$ and the power law $|x|^{2-d}$ for $\chi^{\perp}_{0x}(\beta, 0)$ are taken care of by the 'softening' of the transverse modes as $h \searrow 0$, $\dot{\beta} > \beta_c$, $z(\beta, h) \sim h/m(\beta, 0)$. This is no longer true for the longitudinal fluctuations, where the divergence of $\lambda^{\|}$ as $h \searrow 0$ and the power law $|x|^{2(2-d)}$ for $\chi^{\|}_{0x}(\beta, 0)$ are explained by the non-analyticity which appears at $k = 0$ in $\varphi_z(k)$ as $z \searrow 0$. Maybe unexpectedly, for $d \geqslant 5$, in spite of the fact that the dispersion law still has a 'mass' $2\beta m(\beta, 0)^2\varphi_0(0) > 0$ at $z = 0$, the decay of $\chi^{\|}_{0x}(\beta, 0)$ is not exponential. This behaviour agrees with the predicted one (see [20]), and constitutes a proof of the latter in the large-n limit

(d) Our results complement the heuristic picture for the n-vector model based on the 'spin-wave picture' or the Goldstone one (see for instance in this respect the textbooks by Negele and Orland [21] and Zinn-Justin [22]). They show that indeed the two-point truncated correlation functions have an exponential fall-off with a decay rate proportional to $\sqrt{h}$ on the critical line, far away from the critical point, while at the critical point the truncated correlation functions have a power-law decay, reminding one of the one obtained in asymptotically free massless theories, dictated by the behaviour of $(-\Delta)^{-1}$. Seemingly, the Goldstone picture is contradicted by only one rigorous result obtained for the hierarchical model by Schor and Carroll [23], where the truncated correlation functions parallel to the spontaneous magnetization have a rather strange behaviour. The explanation is that their behaviour is controlled by a non-canonical Gaussian fixed point, which is a property specific to the hierarchical model and is by no means expected to hold in the complete n-vector models.

4. Scaling at the critical temperature

For $h = 0$, $\beta \leqslant \beta_c$, one has $m(\beta, 0) = 0$, $\chi^{\|}_{0x}(\beta, 0) = \chi^{\perp}_{0x}(\beta, 0)$, hence the *large-$x$ behaviour at the critical point* $(\beta_c, 0)$ *and the scaling from the paramagnetic side is taken care of by equations (3.4) and (3.6)*. We shall therefore consider only the scaling of $\chi^{\|}_{0x}(\beta, 0)$ from the ferromagnetic side $\beta \searrow \beta_c$. It turns out that the scaling functions are qualitatively different for $d = 3, 4$ and $d \geqslant 5$.

We take as a parameter $t = \beta - \beta_c$ and denote $\chi^{\|}_{0x}(\beta, 0) = \chi^{\|}(t, x)$.

Proposition 4. *In the sense of definition 1:*

$$\mathcal{D}_d^{\circ\prime} - \lim_{t\searrow 0} t^{-1}\chi^{\|}(t, x/t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\exp \mathrm{i}kx}{\frac{1}{6}k^2 + \frac{4}{9}|k|}\,\mathrm{d}k \qquad \text{if} \quad d = 3 \tag{4.1}$$

$$\mathcal{D}_d^{\circ\prime} - \lim_{t\searrow 0} \frac{|\ln t|}{t}\chi^{\|}\left(t, x\sqrt{\frac{|\ln t|}{t}}\right) = \left(\pi^2/8\right)^{d/2-1} G\left(x\sqrt{\pi^2/8}\right) \qquad \text{if} \quad d = 4 \tag{4.2}$$

$$\mathcal{D}_d^{\circ\prime} - \lim_{t\searrow 0} t^{1-d/2}\chi^{\|}(t, x/\sqrt{t}) = (2\varphi_0(0))^{d/2-1} G(x\sqrt{2\varphi_0(0)}) \qquad \text{if} \quad d \geqslant 5 \tag{4.3}$$

where G is the same function as in proposition 3.

Proof. After applying definition 1 (with parameter t and $\lambda(t)$ as indicated in equations (4.1)–(4.3)) to a certain $\phi \in \mathcal{D}_d^{\circ}$ and going in the left-hand side to Fourier transforms, the result follows by dominated convergence in

$$\lim_{t\searrow 0} \frac{\lambda^{-2}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\hat{\phi}(k)}{\omega(k/\lambda) + 2t\varphi_0(k/\lambda)}\,\mathrm{d}k. \tag{4.4}$$

The integrability at ∞ being ensured by $\hat{\phi}$, one has to care only about the small k behaviour. This follows immediately for $d \geqslant 5$, and for the other cases is accomplished by calculating the following limits: $\lim_{k\to 0} |k|/\varphi_0(k) = \frac{9}{2}$, for $d = 3$ and $\lim_{k\to 0} |\ln|k||\varphi_0(k) = \pi^2/8$, for $d = 4$. $\square$

Remarks

(a) The meaning of the scaling relations in proposition 4 is that it makes sense to speak about a diverging correlation length of the *longitudinal* fluctuations, also when approaching the critical temperature from the low-temperature side, in spite of the power-law decay given by equation (3.5). This can be taken as $\lambda(t)$ of proposition 4, i.e. $1/t$ in $d = 3$, $\sqrt{|\ln t|/t}$ in $d = 4$ and $1/\sqrt{t}$ in $d \geqslant 5$. With this interpretation, the comparison with the high-temperature scaling law, obtained from equation (3.6) and the relation between z and $t < 0$ given by equation (2.21) at $h = 0$, i.e.

$$\lim_{t\nearrow 0} \frac{-t}{z^2} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\tilde{\omega}(k)(1+\tilde{\omega}(k))}\, \mathrm{d}k \qquad (d = 3)$$

$$\lim_{t\nearrow 0} \frac{-t}{z|\ln z|} = \frac{4}{\pi^2} \qquad (d = 4)$$

$$\lim_{t\nearrow 0} \frac{-t}{z} = \frac{1}{\varphi_0(0)} \qquad (d \geqslant 5)$$

shows that *the symmetry of the correlation length critical indices on the two sides of the critical temperature is restored*.

(b) For all $d \geqslant 3$, the homogeneous function $|x|^{2-d}$ appearing in the behaviour of the scaling functions around $x = 0$ has the same order as the one appearing in the large-x decay of the correlations at the critical point, equation (3.4). The fall-off at infinity of the scaling function is exponential for $d \geqslant 4$, but power law for $d = 3$. In the latter case,

$$\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\exp \mathrm{i}kx}{\frac{1}{6}k^2 + \frac{4}{9}|k|}\, \mathrm{d}k = \frac{3}{\pi^2} \frac{\mathrm{si}(8|x|/3)\cos(8|x|/3) - \sin(8|x|/3)\mathrm{ci}(8|x|/3)}{|x|}$$

has the fall-off $|x|^{-2}$ characteristic of the decay of the longitudinal correlations in the ferromagnetic phase, equation (3.5), thus *the scaling function describes the cross-over between the two regimes*, $\beta = \beta_c$ and $\beta > \beta_c$.

(c) The behaviour of the longitudinal correlation functions in three dimensions have also been studied recently by Garanin [24] in the context of the anisotropic spherical model. It has been found that for small wavevectors the longitudinal correlation functions show a non-trivial behaviour in the ordered phase caused by spin-wave fluctuations, reinforcing the spin-wave theory of [15], a behaviour we also found in our study of the isotropic spherical model.

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